Neutrino Oscillations via the Bulk

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Abstract

We investigate the possibility that the large mixing of neutrinos is induced by their large coupling to a five-dimensional bulk neutrino. In the strong coupling limit the model is exactly soluble. It gives rise to an oscillation amplitude whose squared-mass difference is independent of the channel, thus making it impossible to explain both the solar and the atmospheric neutrino oscillations simultaneously.

1 Introduction

The idea of using an extra dimension to explain physics goes back to Kaluza and Klein [1]. Spurred by the discovery of D-branes in string theory [2], this idea was revived in recent years by introducing branes [3] with many variations to the theme. We will adopt here the simplest one, where a single three-brane is embedded in an extra flat dimension. All Standard-Model (SM) particles are confined to our 3+1 dimensional world, the three-brane. Gravity, right-handed neutrinos, and other particles without SM quantum numbers are allowed to live in the 4+1 dimensional bulk. This modification thus introduced to gravity has been tested experimentally [4]. Its consequence on astrophysics and cosmology is extensively investigated [5]. In this paper we will discuss an application of this picture to neutrino physics [6, 7, 8].

If solar and atmospheric neutrino deficits are attributed to neutrino oscillations, then neutrinos must be massive. Direct measurement puts an upper bound of about 2 eV on ν_e [10]. If it is a Majorana particle, the failure to

detect neutrinoless double β decay further limits a combination of the lightest neutrinos and mixings to have a mass less than 0.26 eV [11]. In any case neutrino mass is much smaller than other fermion masses and this mystery should be understood. Moreover, the amount of mixing needed to explain the atmospheric neutrino deficit is large [12], whereas the mixings of quarks are small. Presently these two differences between neutrinos and other fermions are not clearly understood.

In the seesaw mechanism, the smallness of neutrino mass is explained by the high energy scale of the right-handed neutrinos. The smallness can also be attributed to the large size of the extra dimension [6, 7, 8], and it would be nice if the extra dimension can also explain the large neutrino mixing. It is this latter possibility that we carefully examine in the present paper.

To this end we study the simplest model, which consists of one bulk neutrino and three active Majorana brane neutrinos, ν_e, ν_μ, and ν_τ. Since mixings between quarks are small, to simplify matters we shall assume no direct mixings between the active neutrinos in this model. All mixings between them are induced solely through their mass couplings to the bulk neutrino. The bulk neutrino is a Dirac fermion in 4+1 dimensions, and a SM singlet. All in all, the theory contains six parameters, the Majorana masses m_i for the three active neutrinos, and their mass couplings d_i to the bulk neutrino. For equal couplings this model was discussed by Dienes et al in [6]. [Other similar models were also studied [7].] Without Majorana masses and sometimes with only one generation, this model has also been studied by several other groups [7]. The mass couplings d_i modify neutrino masses and induce a mixing between them. If the large mixings seen in experiments are to be explained by this simple model, the mass coupling strengths must be large. We recall that $d_i = y_i v/\sqrt{2\pi M_* R}$ where y_i is the Yukawa coupling between the i-th brane neutrino and the bulk neutrino, v is the usual vacuum expectation value of the SM Higgs field, M_* is the fundamental 5D scale and R is the compactification radius. Hence, the strong coupling case will arise from large brane- bulk coupling and usual perturbative treatments assuming $d_i \ll 1$ will not apply. It is this strong-coupling scenario which we find most interesting and will be the main focus of this paper.

Numerical studies have been carried out in [6, 7] for specific parameters. Our analysis is completely analytical and we find remarkably that there is actually an exact solution to the problem in the strong-coupling limit that we are interested in. In that limit, we are able to give a complete discussion of

the eigenvalues and eigenvectors, as well as exact brane-to-brane transition amplitudes which we have not found in the literature.

In the absence of coupling, the mass eigenvalues are integers for the infinite number of sterile neutrinos composing the bulk neutrino, and m_i for the active neutrinos confined to the brane. All masses are understood to be measured in units of the inverse size of the extra dimension. In the strong-coupling limit, sterile and active components are thoroughly mixed up. The mass eigenvalues now consist of every half-integer, plus two others, λ_b and λ_c, determined by m_i and the fixed ratios of the d_i's. More generally, if there are f active neutrinos then there are $f-1$ such other eigenvalues. With this knowledge, the transition amplitudes from active flavor i to active flavor j $(i,j=\nu_e,\nu_\mu,\nu_\tau)$ can be computed exactly at any 'time' τ. It contains three terms. The first comes from a sum of all transitions via half-integral eigenvalues. It is short lived, and it decays to zero within a short 'time' inversely proportional to the fourth power of the coupling. The second and the third terms are transitions mediated by eigenvalues λ_b and λ_c. After the transient period when the first term dies away, we are left with an amplitude resembling the usual two-neutrino oscillating amplitude, but with a squared-mass difference $\lambda_c^2-\lambda_b^2$ independent of i and j. This independence makes it impossible to explain both the solar neutrino deficit $(i=j=\nu_e)$ and the atmospheric neutrino deficit $(i=j=\nu_\mu)$ simultaneously, for their measured squared-mass differences are at least two orders of magnitudes apart. There are actually also other reasons why this model would not work in practice but they will be discussed later in the main text.

The failure of this model to describe Nature is not due to the presence of sterile neutrinos per se, for they are still allowed. Although sterile neutrinos are disfavored in the solar and atomospheric data of Super-Kamiokande [12], solutions still exist in which the solar oscillation occurs between ν_e and a sterile neutrino [9], and a substantial sterile component is also allowed in atmospheric neutrino oscillations [13]. If LSND experiment is confirmed we do need a sterile component anyway. The failure rather is due to the presence of an infinite number of them, with mass eigenvalues at half integers. It is the regular spacing of these eigenvalues at large coupling that produces a destructive interference at large τ, which eliminates the first term of the transition amplitude, and kills the model. In this our conclusion is consistent with earlier numerical investigations and analytical studies for one active neutrino [6, 7]. If the first term were oscillatory instead of decaying, which

would be the case if the infinite number of them were replaced by a single sterile neutrino, then we would just get a regular three-neutrino oscillation formula and a problem of this nature would not arise.

2 Mass Matrix and Its Diagonalization

We consider a model in which a three-brane is placed at an S^1/Z_2 orbifold fixed point. The size of the extra dimension is measured by the radius R of the circle S^1. The model consists of three left-handed Majorana neutrinos ν_e, ν_μ, and ν_τ, confined to the brane, and a Dirac bulk neutrino Ψ living in the 4+1 dimensional world. The only interactions are mass terms, and we shall express all masses here in units of $1/R$. The bulk neutrino decomposes into an infinite number of 3+1 dimensional sterile neutrinos, with masses $n \in \mathbf{Z}$. The Majorana masses of the brane neutrinos are denoted by m_i, and their couplings to the bulk neutrino are denoted by the complex number d_i. Since phases are not currently detectable, we will assume m_i and d_i to be real. This practice is also adapted in numerical studies [6, 7].

The neutrino mass matrix is given by [6]

$$M = \begin{pmatrix} m_1 & 0 & 0 & d_1 & d_1 & d_1 & d_1 & d_1 & \cdots \\ 0 & m_2 & 0 & d_2 & d_2 & d_2 & d_2 & d_2 & \cdots \\ 0 & 0 & m_3 & d_3 & d_3 & d_3 & d_3 & d_3 & \cdots \\ d_1 & d_2 & d_3 & 0 & 0 & 0 & 0 & 0 & \cdots \\ d_1 & d_2 & d_3 & 0 & 1 & 0 & 0 & 0 & \cdots \\ d_1 & d_2 & d_3 & 0 & 0 & -1 & 0 & 0 & \cdots \\ d_1 & d_2 & d_3 & 0 & 0 & 0 & 2 & 0 & \cdots \\ d_1 & d_2 & d_3 & 0 & 0 & 0 & 0 & -2 & \cdots \\ \cdots & & & & & & \cdots & \cdots & \end{pmatrix}. \tag{1}$$

Let $v = (v_1, v_2, \cdots)^T$ be its column eigenvector with eigenvalue λ. Let $A = \sum_{i=4}^{\infty} v_i$, and $b = \sum_{i=1}^{3} d_i v_i$. The eigenvalue equation $M{\cdot}v = \lambda v$ in component form is

$$\begin{aligned} m_i v_i + d_i A &= \lambda v_i, \qquad (i = 1, 2, 3) &(2)\\ b + c_i v_i &= \lambda v_i, \qquad (i \geq 4) &(3)\end{aligned}$$

where $c_{2n} = 2 - n$ and $c_{2n+1} = n - 1$ for $n \geq 2$. It follows from (3) that

$$\begin{aligned} v_4 &= \frac{b}{\lambda}, \\ v_{2n+3} &= \frac{b}{\lambda + n}, \\ v_{2n+4} &= \frac{b}{\lambda - n}, \qquad (n \geq 1). \end{aligned} \tag{4}$$

Hence

$$A = \sum_{i=4}^{\infty} v_i = b\lambda \sum_{n=-\infty}^{\infty} \frac{1}{\lambda^2 - n^2} = b\frac{\pi}{\tan(\pi\lambda)}. \tag{5}$$

Substituting this into (3), it becomes

$$M'v \equiv \begin{pmatrix} m_1' + d_1^2 & d_1 d_2 & d_1 d_3 \\ d_2 d_1 & m_2' + d_2^2 & d_2 d_3 \\ d_3 d_1 & d_3 d_2 & m_3' + d_3^2 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 0, \tag{6}$$

where

$$m_i' = (m_i - \lambda)\tan(\pi\lambda)/\pi. \tag{7}$$

The eigenvalue λ is then determined by $\det[M'] = 0$. We shall use the characteristic equation in the form

$$\frac{1}{\pi}\tan(\pi\lambda) = \sum_{i=1}^{3} \frac{d_i^2}{\lambda - m_i}. \tag{8}$$

2.1 Eigenvalues and Eigenvectors

The eigenvalues can be determined in the following way. In the unit interval $(n - \frac{1}{2}, n + \frac{1}{2})$, $n \in \mathbf{Z}$, the left-hand side of (8) increases from $-\infty$ to $+\infty$ as λ moves from the left edge to the right edge of this interval. On the other hand, the function on the right-hand side of (8) is a monotonically decreasing function of λ in each of the four intervals $(-\infty, m_1), (m_1, m_2), (m_2, m_3), (m_3, +\infty)$, if $m_1 < m_2 < m_3$. It decreases from 0 to $-\infty$ in the first interval, from $+\infty$ to $-\infty$ in the second and third intervals, and from $+\infty$ to 0 in the fourth

interval. Hence if we divide the real axis into intervals bounded by half integers (odd integers divided by 2), as well as m_1, m_2, m_3, then there is one and only one solution of (8) in each of these intervals. See Fig. 1.

From (6), we can determine (v_1, v_2, v_3) simply by taking the cross product of any two rows of the matrix M'. Using (7) and (8), the result (to within an arbitary normalization) can be taken to be

$$v_i = \frac{d_i}{\lambda - m_i}, \qquad (i = 1, 2, 3). \tag{9}$$

With this normalization,

$$b = \sum_{i=1}^{3} d_i v_i = \sum_{i=1}^{3} \frac{d_i^2}{\lambda - m_i} = \frac{1}{\pi}\tan(\pi\lambda), \tag{10}$$

where (8) is used in the last expression. The rest of the components, v_i for $i \geq 4$, is determined by (4). The norm of the vector v is equal to

$$B^2 \;=\; v\cdot v = \sum_{i=1}^{3} \left(\frac{d_i}{\lambda - m_i}\right)^2 + \frac{1}{\cos^2(\pi\lambda)}. \tag{11}$$

The second term in (11) is obtained from

$$B_\perp^2 \;\equiv\; \sum_{i=4}^{\infty} v_i^2 = b^2 \sum_{n=-\infty}^{\infty} \frac{\lambda^2 + n^2}{(\lambda^2 - n^2)^2} = \left(\frac{\pi b}{\sin(\pi\lambda)}\right)^2 \tag{12}$$

by using (10).

2.2 Unitary matrix

It is more convenient to replace the flavor indices $i = 1, 2, 3$ by a_1, a_2, a_3. Index $i = 2n + 3$ for $n \geq 1$ will be renamed index $-n$, and index $i = 2n + 4$ for $n \geq 0$ will be renamed index n. With this alteration, eqs. (9) and (4) become

$$\begin{aligned} v_{a_i} &= \frac{d_i}{\lambda - m_i}, \qquad (a_1, a_2, a_3) = (\nu_e, \nu_\mu, \nu_\tau) \\ v_n &= \frac{b}{\lambda - n}, \qquad (n \in \mathbf{Z}). \end{aligned} \tag{13}$$

We shall refer to a_i as the brane components, and n as the bulk components.

The overlapping matrix between a normalized eigenvector $|\lambda\rangle$ and a flavor component $\langle f|$ ($f = a_i, n$) is then given by the unitary matrix

$$U_{f\lambda} = \langle f|\lambda\rangle = \frac{v_f(\lambda)}{B(\lambda)}, \tag{14}$$

where $v_f(\lambda)$ is given by (13) and $B(\lambda)$ is obtained from (11).

3 Neutrino Oscillations

The probability for a brane component a_j to turn into a brane component a_i, after the neutrino of energy E has travelled a distance L,

$$\mathcal{P}_{ij}(\tau) = \Big|\mathcal{A}_{ij}(\tau)\Big|^2, \tag{15}$$

is given in terms of the transition amplitude

$$\mathcal{A}_{ij}(\tau) = \sum_{\lambda} U^*_{a_i\lambda} U_{a_j\lambda} e^{-i\lambda^2\tau}, \tag{16}$$

where $\tau = L/2E$ for vacuum oscillations. The sum over eigenvalues λ is an infinite sum and can be evaluated only when the eigenvalues are explicitly known.

4 Strong Coupling Limit

When $d_i = 0$, the neutrino mass matrix M is diagonal. The eigenvalues are m_i for the brane states of active neutrinos, and $n \in \mathbf{Z}$ for the bulk states of sterile neutrinos. There is no mixing between any two flavor states.

When $d_i \neq 0$, eigenvalues shift, and mixing between flavor states are induced. If d_i are small, the shifts and mixings can be calculated by perturbation theory. This however is not the interesting regime because it will never generate the observed neutrino flavor mixings, which are large.

When d_i are big, there is a chance that such large mixings can be induced. This is the regime we will study in the present section. Remarkably, in this strong coupling regime, the eigenvalues can be obtained explicitly and the infinite sum encountered in (16) can be computed.

4.1 Eigenvalues

Let $d_i = de_i$, $\sum_{i=1}^{3} e_i^2 = 1$, and $d \gg 1$. For fixed e_i and increasing d, the right-hand side of (8) increases in magnitude, so the left-hand side must change accordingly. If the equality in (8) is positive, the eigenvalue λ must move to the right (R). If it is negative, λ must move to the left (L).

Divide the real axis into six sections, $J_1 = (-\infty, m_1),\ J_2 = (m_1, \lambda_b),\ J_3 = (\lambda_b, m_2),\ J_4 = (m_2, \lambda_c),\ J_5 = (\lambda_c, m_3),\ J_6 = (m_3, \infty)$, where λ_b and λ_c are the two zeros of

$$r(\lambda) \equiv \sum_{i=1}^{3} \frac{e_i^2}{\lambda - m_i}, \tag{17}$$

explicitly given by

$$\begin{aligned} \lambda_{b,c} &= \frac{1}{2}(p \mp c), \\ p &= e_1^2(m_2 + m_3) + e_2^2(m_3 + m_1) + e_3^2(m_1 + m_2), \\ c &= \left[p^2 - 4(e_1^2 m_2 m_3 + e_2^2 m_3 m_1 + e_3^2 m_1 m_2)\right]^{\frac{1}{2}}. \end{aligned} \tag{18}$$

Then λ moves L if it is in $J_{1,3,5}$, and λ moves R if it is in $J_{2,4,6}$. See Fig. 1.

Remember that if the real axis is divided into intervals I_λ bounded by the half integers and m_i, then there is one and only one eigenvalue λ in each interval. If $I_\lambda \subset J_{1,3,5}$, then λ moves L. If $I_\lambda \subset J_{2,4,6}$, then λ moves R. If I_λ contains either λ_b or λ_c, then λ moves towards these zeros of $r(\lambda)$. In particular, if m_i is one of the two boundaries of I_λ, then λ always moves away from that boundary. If both boundaries are some m_i, then a zero $\lambda_{b,c}$ must be contained in that interval where the eigenvalue moves towards. In any case, the eigenvalue always stays away from the m_i boundaries.

As $d \to \infty$, the eigenvalues will therefore end up either at the half-integer boundaries, or a zero λ_b, or λ_c. In fact, there is one and only one eigenvalue at every half integer.

To see that, choose any half integer, and denote the interval to its left (right) by I_- (I_+). The only way this half integer ends up not to be an eigenvalue, or end up to be equal to two different eigenvalues, is when the two λ's in I_- and I_+ move in opposite directions as d increases. This however is not possible unless a zero is contained in I_- or in I_+. If it is in I_-, then $\lambda \in I_-$ will drift towards this zero, and $\lambda \in I_+$ will move L, drifting towards

the half-integer value in question. Similar thing happens when the zero is in I_+. We therefore conclude that there is one and only eigenvalue at each half integer value.

It is important to note that the eigenvalues λ_b and λ_c are special simply because they do not fall into this infinite sequence of half integers. It is not because they are brane states. In the strong coupling limit, brane and bulk components are thoroughly mixed up. Nevertheless, we may still define a brane state as one that extrapolates back to the eigenvalues m_i when $d_i = 0$, and a bulk state as one that extrapolates back to an integer eigenvalue in that limit. With that definition, each zero may correspond to either a brane state or a bulk state. If I_λ containing this zero also contains an integer, then it is a bulk state. If it does not, then it must be a brane state.

4.2 Transition amplitude

With this strong-coupling spectrum, the transition amplitude in (16) can be written more explicitly as

$$\begin{aligned}\mathcal{A}_{ij}(\tau) &= d^2 e_i e_j \Bigg[\sum_{n=-\infty}^{\infty} \left(\frac{e^{-i\lambda_n^2 \tau}}{B^2(\lambda_n)(\lambda_n - m_i)(\lambda_n - m_j)} \right) \\ &\quad + \frac{e^{-i\lambda_b^2 \tau}}{B^2(\lambda_b)(\lambda_b - m_i)(\lambda_b - m_j)} \\ &\quad + \frac{e^{-i\lambda_c^2 \tau}}{B^2(\lambda_c)(\lambda_c - m_i)(\lambda_c - m_j)} \Bigg], \end{aligned} \tag{19}$$

For $d \gg 1$, there is a qualitative difference between the normalization factors $B^2(\lambda_n)$ and $B^2(\lambda_{b,c})$. This is so because the right-hand side of (8), which according to (17) can be written as $d^2 r(\lambda)$, behaves differently for the two sets of eigenvalues. For $\lambda_{b,c}$, $r(\lambda) = O(1/d^2)$ so these eigenvalues as well as $\cos(\pi\lambda)$ are finite in the limit $d \to \infty$. For λ_n, $r(\lambda) = O(d^2/(\lambda - m_i))$, so in the strong coupling limit $\tan(\pi\lambda)$ as well as $1/cos(\pi\lambda)$ are of order d^2 for finite λ, and are of order $O(d^2/\lambda)$ when λ is comparable or larger than d^2. Using (11), it follows then that $B^2(\lambda)$ is of order d^2 for $\lambda_{b,c}$, but is of order d^4/λ for λ_n.

Returning to (19), it follows from this observation that the second and third terms on the right are finite in the strong coupling limit, but each of

the terms in the infinite sum is of order $1/d^2$ when $\lambda_n \ll d^2$. This means that we can freely drop a finite number (say N) of terms from the infinite sum in the strong coupling limit, and retains only those terms in the infinite sum with large λ_n. In that case, the right-hand side of (8) becomes d^2/λ, and we can use it to solve for $1/\cos^2(\pi\lambda)$ to get $B^2(\lambda) \simeq 1/\cos^2(\pi\lambda) = 1 + d^2(1+\pi^2 d^2)/\lambda^2 \equiv 1 + K^2/\lambda^2$. Since $\Delta\lambda_n \ll d^2$ and $N/d^2 \ll 1$, we may now replace the infinite sum on the right of (19) by the integral

$$f(\tau) \;=\; \int_{-\infty}^{\infty} d\lambda \frac{e^{-i\lambda^2\tau}}{\lambda^2+K^2}. \qquad (20)$$

At $\tau = 0$, the integral can be evaluated to give $f(0) = \pi/K \simeq 1/d^2$.

Moreover, for $d \gg 1$, it follows from (11) that $B^2(\lambda) \simeq d^2 s(\lambda)$ is true for $\lambda = \lambda_b, \lambda_c$ (assuming that $\cos(\lambda_{b,c}) \neq 0$), where

$$s(\lambda) \equiv \sum_{i=1}^{3} \left(\frac{e_i}{\lambda - m_i}\right)^2. \qquad (21)$$

Therefore,

$$\begin{aligned} \mathcal{A}_{ij}(0) \;=\; e_i e_j \Bigg[1 + &\frac{1}{s(\lambda_b)(\lambda_b - m_i)(\lambda_b - m_j)} \\ &+ \frac{1}{s(\lambda_c)(\lambda_c - m_i)(\lambda_c - m_j)} \Bigg]. \end{aligned} \qquad (22)$$

According to (16), this should be equal to δ_{ij} for all parameters e_i and m_i, if the infinite-dimensional matrix $U_{f\lambda}$ is unitary, as expected from the completeness relation. This is indeed so and can be verified numerically. To prove the identity analytically, a change of parameters is desirable.

The model is specified by six parameters: $d_i = de_i$ and m_i. They are actually not the most convenient parameters to use because according to (8), a shift of the m_i's by a common integer simply shifts all the eigenvalues by the same integer, so the more useful parameters are $\lambda - m_i$ rather than m_i. In fact, all the quantities in (22) can be expressed in terms of the parameters

$$x_i = \frac{1}{\lambda_b - m_i}, \quad y_i = \frac{1}{\lambda_c - m_i}. \qquad (23)$$

Note that these two sets of parameters are related by

$$\frac{1}{y_i} - \frac{1}{x_i} = \lambda_c - \lambda_b = c, \tag{24}$$

so that there are only four independent parameters in the set: c and x_i, or c and y_i.

We can use the three conditions, $e^2 = 1$ and $r(\lambda_b) = r(\lambda_c) = 0$, to solve for the three variables e_1^2, e_2^2, e_3^2. The result is

$$\begin{aligned} e_1^2 &= (x_2 y_3 - y_2 x_3)/D, \\ e_2^2 &= (x_3 y_1 - y_3 x_1)/D, \\ e_3^2 &= (x_1 y_2 - y_1 x_2)/D, \\ D &= (x_1 y_2 + x_2 y_3 + x_3 y_1) - (y_1 x_2 + y_2 x_3 + y_3 x_1). \end{aligned} \tag{25}$$

We can use (24) to eliminate either y_i or x_i, and we get respectively

$$s_b \equiv s(\lambda_b) = c x_1 x_2 x_3, \tag{26}$$

or

$$s_c \equiv s(\lambda_c) = -c y_1 y_2 y_3 = -c \frac{x_1 x_2 x_3}{(1 + c x_1)(1 + c x_2)(1 + c x_3)}. \tag{27}$$

In these notations, we can write (22) in a more transparent form as

$$\mathcal{A}_{ij}(0) \;=\; \sum_{p=1}^{3} V_{ip}^* V_{jp}, \tag{28}$$

where

$$V \;=\; \begin{pmatrix} e_1 & e_1 x_1/\sqrt{s_b} & e_1 y_1/\sqrt{s_c} \\ e_2 & e_2 x_2/\sqrt{s_b} & e_2 y_2/\sqrt{s_c} \\ e_3 & e_3 x_3/\sqrt{s_b} & e_3 y_3/\sqrt{s_c} \end{pmatrix}, \tag{29}$$

up to an arbitrary phase for each column. The matrix V is unitary, as can be verified from the explicit equations (24) to (27).

The matrix V is parametrized by four real parameters, c, x_i, plus three arbitrary phases of e_i, more than the usual 3×3 mixing matrix. Nevertheless, there are mixing matrices not of the form (29), with the constraint (24) to

(27). An example is the bimaximal neutrino mixing matrix ($s_3 = \sin\theta_3$, $c_3 = \cos\theta_3$):

$$u = \begin{pmatrix} c_3/\sqrt{2} & c_3/\sqrt{2} & s_3 \\ -\frac{1}{2}(1+s_3) & \frac{1}{2}(1-s_3) & c_3/\sqrt{2} \\ \frac{1}{2}(1-s_3) & -\frac{1}{2}(1+s_3) & c_3/\sqrt{2} \end{pmatrix}. \tag{30}$$

We are now ready to discuss the case $\tau > 0$. The function $f(\tau)$ can be expressed in terms of error function with an imaginary argument, but it is more straight forward to write it directly in the form $f(\tau) = (\pi/K)g(K^2\tau)$, where

$$g(x) \equiv \frac{1}{\pi}\int_{-\infty}^{\infty} du \frac{e^{-iu^2x}}{u^2+1}. \tag{31}$$

The absolute magnitude of the function $g(x)$ decreases monotonically from 1 to zero. For large x, $g(x) \simeq (1-i)/\sqrt{2\pi x}$.

The transition amplitude (19), in the strong coupling limit $d \gg 1$, is therefore equal to

$$\begin{aligned}\mathcal{A}_{ij}(\tau) &= e_ie_j\Bigg[g(K^2\tau) + \frac{e^{-i\lambda_b^2\tau}}{s(\lambda_b)(\lambda_b-m_i)(\lambda_b-m_j)} \\ &\qquad + \frac{e^{-i\lambda_c^2\tau}}{s(\lambda_c)(\lambda_c-m_i)(\lambda_c-m_j)}\Bigg] \\ &= V_{i1}^*V_{j1}g(K^2\tau) + V_{i2}^*V_{j2}e^{-i\lambda_b^2\tau} + V_{i3}^*V_{j3}e^{-i\lambda_c^2\tau}, \end{aligned} \tag{32}$$

with the matrix V given in (29). At any 'time' τ, the amplitude is described by six parameters: K, c, x_1, x_2, x_3, and one of the m_i's so that λ_b and λ_c can be determined from x_i and y_i respectively. Alternatively, we can let λ_b be the sixth parameter, whence $\lambda_c = \lambda_b + c$.

4.3 Comparison with direct mixing

In this theory, neutrino mixing is induced by coupling of active neutrinos to bulk neutrinos. No direct mixing of active neutrinos takes place. Nevertheless, in the strong coupling limit, the resulting amplitude (32) bears a close resemblance to the usual theory in which sterile neutrino is absent, and mixing of active neutrinos is directly introduced through the 3×3 MNS matrix

u [14]. If μ_1, μ_2, μ_3 are the mass eigenstates, the transition amplitude with direct mixing is given by

$$\mathcal{A}_{ij}(\tau) \;\; = \;\; u_{i1}^* u_{j1} e^{-i\mu_1^2\tau} + u_{i2}^* u_{j2} e^{-i\mu_2^2\tau} + u_{i3}^* u_{j3} e^{-i\mu_3^2\tau}. \tag{33}$$

Since both V and u are unitary matrices, if we identify λ_b^2 with μ_2^2 and λ_c^2 with μ_3^2 (modulo a common constant), then the main difference between (32) and (33) is the replacement of $g(K^2\tau)$ by $e^{-i\mu_1^2\tau}$. Both functions equal to unity at $\tau = 0$, but the latter oscillates and the former decays to zero within a short transient 'time' $\sim 1/K^2 \ll 1$. The oscillations mediated by three mass eigenstates in the direct-mixing theory, is replaced after the transient period by an oscillation mediated by two eigenstates (λ_b and λ_c) in the bulk theory. Note that these two eigenstates can be either brane states or bulk states (see the discussion at the end of Sec. IIIA), so in general they have no direct connection with the three mass eigenstates in the direct-mixing theory. In particular, we should not think that one of these three mass eigenstates gets lost after the transient period.

Moreover, unlike the direct-mixing theory in which the total probability to be in some active neutrino state is always 1, in the bulk theory this total probability decreases with τ because of leakages into the bulk. Quantitatively,

$$\mathcal{P}_j(\tau) \equiv \sum_{i=1}^{3} \mathcal{P}_{ij}(\tau) = 1 - e_j^2\{1 - |g(K^2\tau)|^2\} \simeq 1 - e_j^2 \quad (\tau \gg K^{-2}). \tag{34}$$

As a check we go to the limit of equal couplings, i.e. all e_i's are equal. Since $\sum_i e_i^2 = 1$ then each $e_i^2 = 1/3$ and the above probability approached 2/3 in agreement with the result of [6].

In the bulk neutrino theory, with an infinite number of sterile neutrinos, is very different from the direct-mixing theory with three active neutrinos, or three active neutrino plus one sterile neutrino. In particular, there is no way the bulk neutrino theory can explain both the solar and the atmospheric neutrino deficits simultaneously, at least not in the strong-coupling limit after the transient period is passed.

To see that, we use (32) to write the survival probability after the transient period in the form

$$\mathcal{P}_{ii}(\tau) \;\; = \;\; |\mathcal{A}_{ii}(\tau)|^2 = p_i - p_i' \cos^2\left(\frac{1}{2}\Delta^2\tau\right),$$

$$
\begin{aligned}
p_i &= \left(|V_{i2}|^2 + |V_{i3}|\right)^2 = (1-e_i)^2\,, \\
p_i' &= 4(1-e_i^2)|V_{i3}|^2, \\
\Delta^2 &= \lambda_c^2 - \lambda_b^2.
\end{aligned} \tag{35}
$$

This is to be compared with the usual two-term survival probability formula for solar and atmospheric neutrinos,

$$
\mathcal{P}_i(\tau) \;=\; 1 - \sin^2(2\theta_i)\sin^2\left(\frac{1}{2}\Delta_i^2\tau\right), \tag{36}
$$

where Δ_1^2 for solar neutrino oscillations is at least two orders of magnitude smaller than Δ_2^2 for atmospheric neutrino oscillations. There is no way the formula (35), with Δ independent of i, can fit both the solar and the atmospheric deficits.

We might want to be less ambitious and use (35) to fit only solar (or atmospheric, but not both) neutrino deficit by adjusting the parameters for (35) to emulate (36) with the experimentally measured θ_i and $\Delta_i = \Delta$. Even so it will not work, not exactly anyway. The reason is that to make $p_i = 1$, we must put $e_i = 0$. In that case (17) has only one but not two zeros, and the three terms appearing in (32) become two terms. After the transient period, we are left with one term, whose probability does not oscillate at all.

These failures are due to the disappearance of the first term beyond the transient period. What about taking the opposite limit when τ is so small that $g(K^2\tau) \simeq 1$? That would not work either for two reasons. First, in order for oscillations to occur we must have $\lambda_b^2\tau$ and/or $\lambda_c^2\tau$ to be of order 1, which requires $\lambda_b^2 \gg K^2$ and/or $\lambda_c^2 \gg K^2$. This cannot be achieved within the strong coupling limit when e_i and m_i are fixed and $d \gg 1$. Moreover, even if we ignore this difficulty, there is no way the unitary matrix V can simulate a realistic MNS matrix u, as pointed out previously in (30).

In conclusion, the idea of using a bulk neutrino living in an extra dimension to explain neutrino oscillation is a very attractive one. It might explain why neutrino masses are small and in the strong brane-bulk coupling limit, why their mixings are big. Unfortunately, such strong coupling also causes a large leakage into the sterile modes in the bulk, so that this theory in the strong-coupling limit is not compatible with experiments. Whether an intermediate coupling is viable or not depends on whether rapid variations as a function of energy and distance are observed in future experiments.

Our study complements the mostly numerical work done on this simple model. Since the solution is nonperturbative, we deem this work to be an important first in the study of the physics of strongly coupled bulk neutrinos. As more data in solar and atmospheric neutrinos become available there will be more serious attempts at seeing whether bulk neutrinos can explain all or part of such improved data sets. Our work can be used as a guide to such attempts.

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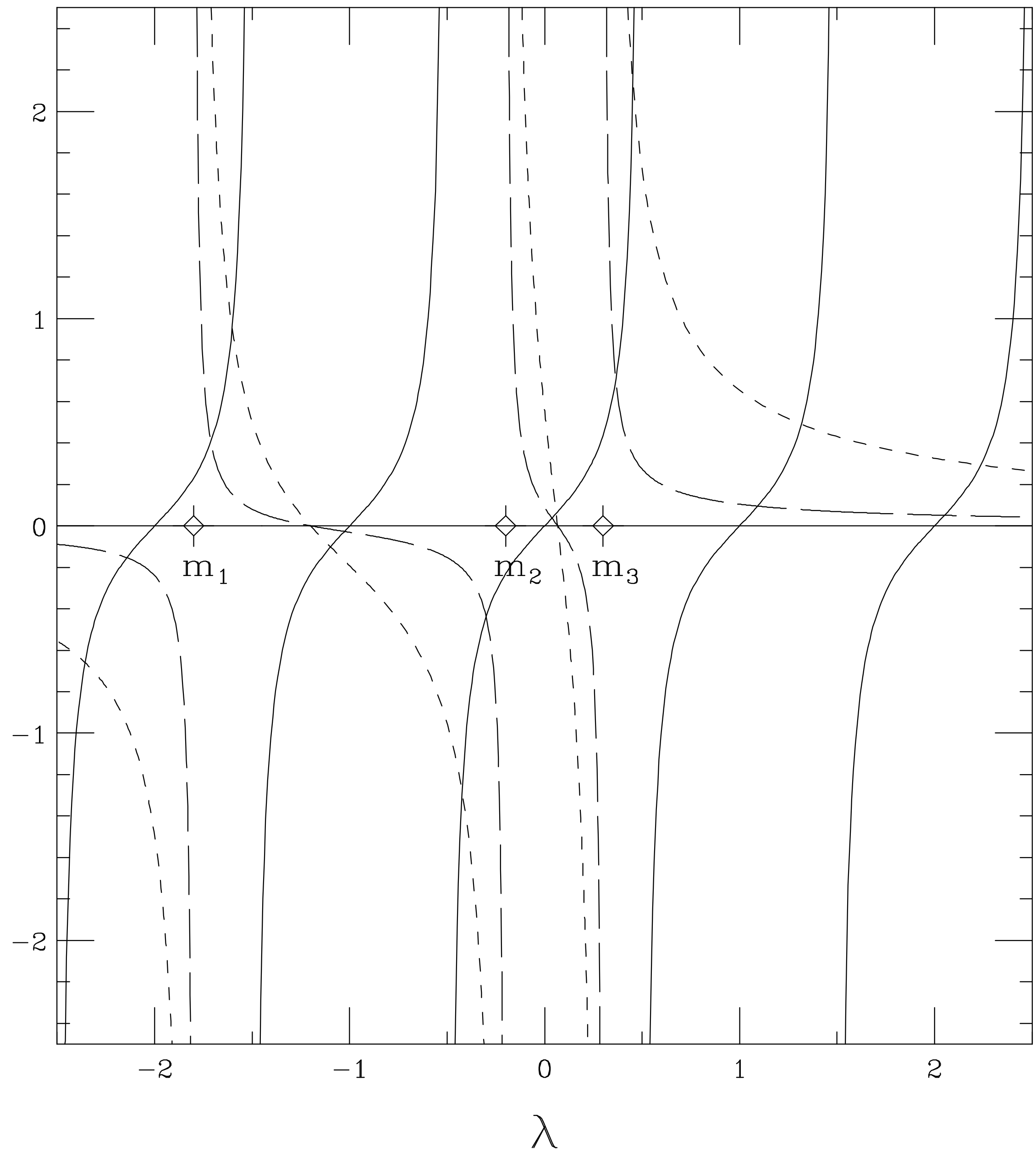

Figure 1: Graphical solutions of the characteristic equation (8). The left hand side of (8) is represented by the solid curve; the right hand side is represented by the dashed curves, with $(m_1, m_2, m_3) = (-1.8, -0.2, 0.3)$. The one with the short dash has the larger coupling constant d than the one with the long dash. Note that the solution λ moves up and right along the solid curve for increasing d when the value on both sides of (8) is positive, and moves down and left along the solid curve when the value is negative.